

Endomorphic Elements in Banach Algebras

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Abstract

The use of the properties of actions on an algebra to enrich the study of the algebra is well-trodden and still fashionable. Here, the notion and study of endomorphic elements of (Banach) algebras are introduced. This study is initiated, in the hope that it will open up, further, the structure of (Banach) algebras in general, enrich the study of endomorphisms and provide examples.

In particular, here, we use it to classify algebras for the convenience of our study. We also present results on the structure of some classes of endomorphic elements and bring out the contrast with idempotents.

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1 Motivation

Dictated by convenience, we shall introduce a twist in the usual terminologies concerning elements. Let A be a complex or real algebra. $a \in A$ will be called a left element, a right element or just an element according as a is, considered as a self map of A , of the form $x \mapsto ax$, $x \mapsto xa$, or $x \mapsto axa$. A bi-element is a left and right element. A self map T of A is said to be endomorphic if it is linear and $T(ab) = T(a)T(b) \quad \forall a, b \in A$. The meaning of terms like endomorphic left element, compact endomorphic right element, etc becomes immediately obvious.

The disc algebra is a semisimple commutative Banach algebra having its maximal ideal space homeomorphic with $(0,1)$. It has no nontrivial endomorphic map. In 1980 Kamowitz [4] posed the problem of studying compact endomorphic maps of Banach algebras in general to see if the situation generalizes. In the above paper and again in 1989 [5], 1998 [6], 2000 [2], and 2004 [3], along with others, he went on to discuss this for commutative semisimple algebras. To complement their work, I have relaxed the restriction on the algebra to accommodate all algebras and considered endomorphic elements in place of compact endomorphic maps. This is with a view of throwing more light on the study of endomorphic maps in general.

Another motivation for this attempt is a paper of Zemanek [7] on idempotents which he presented to the International Congress of Mathematicians in Helzinki in 1978. There he discussed the structure of idempotents in Banach algebras in general and semisimple Banach algebras in particular. That endomorphic elements and idempotents are closely related is emphasized and exploited in our presentation.

More particularly, this paper deals with the structure of $L(A)$, the class of endomorphic left elements in A given by

$$L(A) := \{a \in A : axay = axy \quad \forall x, y \in A\}.$$

Other possibility is the class $R(A)$ of all endomorphic right elements of A . This class is by implication already being treated as appropriate adjustments of the $L(A)$ -situation will cover this.

Unexplained terms are those of [1]. Except otherwise explained, A is an algebra.

2 Introduction of Appropriate Terms and Notations

2.1 Definition Let A be an algebra. Then $I(A)$ denotes the class of all idempotents of A .

2.2 Definition

(1) *Algebra Without Order*: An algebra A is without order iff

$$(a \in A) \wedge [(ax = 0 \ \forall x \in A) \vee (xa = 0 \ \forall x \in A)] \Rightarrow (a = 0).$$

(2) *Nice Algebra*: An algebra A is nice iff

$$(a \in A) \wedge (axy = axay \ \forall x, y \in A) \Rightarrow (ax = axa \ \forall x \in A).$$

(3) *Very Nice Algebra*: An algebra A is very nice iff

$$(a \in A) \wedge (axy = axay \ \forall x, y \in A) \Rightarrow (a \in I(A)) \wedge (ax = axa \ \forall x \in A).$$

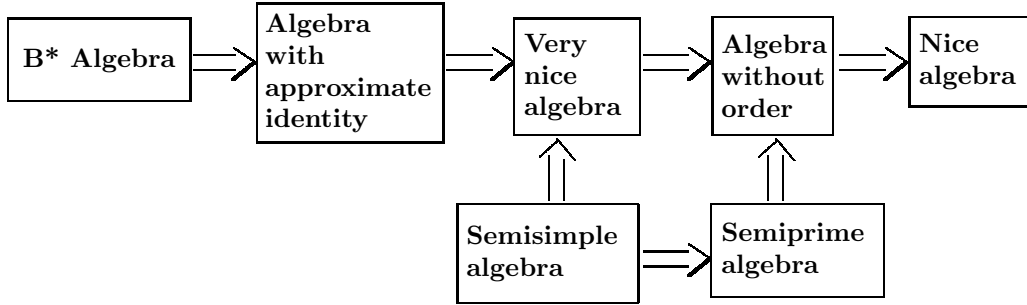


Figure 1: Relations Among Relevant Types of Algebras

2.3 Definition

(1) $N'_3(A)$ denotes the maximal nilpotent subalgebra of the algebra A of order 3. i.e.

$$N'_3(A) := \{a \in A : axy = 0 \ \forall x, y \in A\}.$$

(2) $N_3(A) := \{a \in A : a^3 = 0\}$.

(3) $N(A) := \{a \in A : a^n = 0 \text{ for some } n \in \mathbb{N} := \{1, 2, 3, \dots, n, \dots\}\}$.

(4) If A is a normed algebra, then $QN(A)$ denotes the class of all quasinilpotent elements of A .

2.4 Definition A is an endomorphic left algebra iff $A = L(A)$.

2.5 Definition If A is a normed algebra and $U, V \in A$ then

$$d(U, V) := \inf_{x \in U, y \in V} \|x - y\|.$$

3 General Properties of Endomorphic Left Elements

3.1 Theorem The following statements hold:

(1) $(a \in L(A)) \Rightarrow (\exists n \in \mathbb{N} \text{ such that } 1 \leq n \leq 3 \text{ and } a^n = a^{n+1})$.

(2) $L(A)I(A) \subset I(A)$.

(3) $[L(A)]^2 \subset L(A)$.

$$(4) L(A) = \bigcup_{b \in G(\tilde{A})} bL(A)b^{-1}$$

where $G(\tilde{A})$ is the class of all invertible elements of \tilde{A} , the minimal algebra which has identity and contains A .

(5) If A is a normed algebra then $L(A)$ is closed in A .

Proof (1) – (4) follow easily from the definition of a endomorphic left element and manipulation of standard techniques. Take, for example, (3) and (5).

Consider (3). Let $a, b \in L(A)$. Then

$$abxaby = abxay = abxy \quad \forall x, y \in A.$$

Therefore $ab \in L(A)$ and (3) holds.

That $L(A)$ is closed in A follows from the joint continuity of product in A . \square

3.2 Note

(1) If $(a \in L(A)) \wedge (a \neq 0) \wedge (a \neq 1)$ then a is a divisor of zero.

(2) An immediate conclusion from (1) is the following: An algebra without a divisor of zero has no non-trivial endomorphic elements.

(3) The inclusion in 3.1(3) can be strict. Take a non-trivial nilpotent algebra A of order 2. Then $A = L(A)$ and $[L(A)]^2 = \{0\}$.

(4) If A is nice then

(i) $(a \in L(A)) \Rightarrow (\exists n \in \mathbb{N} \text{ such that } 1 \leq n \leq 2 \text{ and } a^n = a^{n+1})$.

(ii) $[L(A)]^2 = L(A) \cap I(A)$.

Reason. (i) follows from the definition of a nice algebra. Consider (ii). From 3.1(3) and the definition of a nice algebra, $[L(A)]^2 \subset L(A) \cap I(A)$. Take $a \in I(A) \cap L(A)$. Then $a^2 = a$. Therefore $a \in [L(A)]^2$. Therefore $[L(A)]^2 \supset L(A) \cap I(A)$.

(5) Part of Zemanek's characterization in [7] of central idempotents is that for a semisimple Banach algebra A

$$(e \in Z(A) \cap I(A)) \Leftrightarrow (eI(A) \subset I(A)).$$

There He also gave an example of a non-semisimple Banach algebra having a non-central idempotent e with

$$eI(A) \subset I(A).$$

In fact his A is the Banach algebra of complex upper triangular 2×2 matrices $\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$ and his e is $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Indeed, this observation is part of a greater truth which is 3.1(2).

(6) Equality holds in 3.1(2) if $I(A) \subset Z(A)$.

Reason. In this case $I(A) \subset L(A)$ and $a \in I(A) \subset L(A) \Rightarrow a = aa \in L(A)I(A)$. Thus

$$L(A)I(A) \supset I(A).$$

(7) Interpreting (6) gives the following: If in an algebra A every idempotent is in the centre of A then every endomorphic left element can be factored as the product, ai , of an endomorphic left element and an idempotent.

4 Nilpotency and Endomorphic Left Elements

4.1 Theorem The following statements are equivalent:

(1) $a \in N(A) \cap L(A)$.

(2) $a \in N_3(A) \cap L(A)$.

(3)] $a \in N'_3(A)$.

If A is a normed algebra, then each of (1) – (3) is equivalent to the following:

(4) $a \in QN(A) \cap L(A)$.

Proof

$$\begin{aligned} QN(A) \cap L(A) &\subset N(A) \cap L(A) \subset N_3(A) \cap L(A) \\ &\subset N'_3(A) \subset N(A) \cap L(A) \subset QN(A) \cap L(A) \end{aligned}$$

is immediate from definition and 3.1(1). The first and last terms are included only if A is a normed algebra. \square

4.2 Theorem A is an endomorphic left algebra iff $A = N'_3(A)$. i.e. $A^3 = \{0\}$.

Proof Let K be the scalar field of A . Let $\alpha \in K$ be arbitrary. Then

$$\begin{aligned} (A = L(A)) \wedge (\exists b \in A \text{ with } b^3 \neq 0) \\ \Rightarrow (b^4 = b^3) \wedge ((\alpha b)^4 = (\alpha b^3)), \quad \text{from 3.1} \\ \Rightarrow (\|b^4\| = \|b^3\|) \wedge (|\alpha|^4 \|b^4\| = |\alpha|^3 \|b^3\|) \\ \Rightarrow \alpha = \pm 1 \text{ or } 0. \end{aligned}$$

This is a contradiction since $\alpha \in K$ is arbitrary. Hence

$$(L(A) = A) \Rightarrow (a^3 = 0 \quad \forall a \in A).$$

The reverse implication follows from 4.1. \square

4.3 Theorem The following statements are equivalent:

(i) $N'_3(A) = \{0\}$.

(ii) A is without order.

Proof It is easy to see. \square

4.4 Corollary Let A be an algebra without order and an endomorphic left algebra. Then A is trivial.

4.5 Note If A in Theorems 4.1 – 4.3 is in addition nice then the suffix and superfix 3 can be replaced by 2 everywhere they appear in these theorems. Also in the sequel, all general results involving suffix 3 / superfix 3 are also true when the algebra is, in addition, assumed nice and the said 3 is replaced by 2. All these can be seen with the help of 3.2(4)(i), the observation that for a nice algebra A ,

$$N'_3(A) = N'_2(A), \quad N_3(A) = N_2(A)$$

and some manipulation. \square

5 Connectedness of $L(A)$, A , a Banach Algebra

Throughout this section, A is, among other things, a Banach algebra. We continue with our structure theorems under this additional assumption.

5.1 Theorem Let A , be a Banach algebra. Then the following hold:

(1) $N'_3(A)$ is the component of the origin in $L(A)$.

(2) $N'_3(A)$ is unbounded if it is not a singleton.

(3) $L(A)$ is disconnected iff

$$Q(A) := L(A) \setminus N'_3(A) \neq \emptyset.$$

Moreover,

$$(4) \quad Q(A) \neq \emptyset \Rightarrow d(N'_3(A), Q(A)) \geq 1.$$

Proof Consider (1). $N'_3(A) \subset C_0$, the component of $L(A)$ containing the origin. This is because

$$(0 \neq a \in N'_3(A)) \Rightarrow (f := t \mapsto ta : [0, 1] \rightarrow N'_3(A)).$$

Thereby joining the origin to a through $N'_3(A)$. To establish $C_0 = N'_3(A)$ it remains to prove (i). Suppose $a \in N'_3(A)$ and $b \in Q(A)$. Then $(b-a)b^3 = b^3$ and so $\|b-a\| \geq 1$. Therefore $d(N'_3(A), Q(A)) \geq 1$. Hence (1) holds.

Consider (2). If $N'_3(A)$ is not a singleton then $\exists a \in N'_3(A)$ such that $a \neq 0$. Then $\{ta : t \in [0, \infty)\}$ is an unbounded subset of $N'_3(A)$, thereby making $N'_3(A)$, itself, unbounded.

(3) will be established, in view of (4), if we show that:

$$(L(A) \text{ is disconnected}) \Rightarrow (L(A) \text{ has a non-quasinilpotent member}).$$

Now its contrapositive is:

$$(L(A) = N'_3(A)) \Rightarrow (L(A) \text{ is connected})$$

which is true by (1). □

Our next main task is to characterize the isolated points of $L(A)$. To facilitate this we establish the following lemma.

5.2 Lemma Suppose $a, b \in L(A)$ with $ab = ba$, $b^3 \neq a$ and $a \in I(A)$. Then $\|a - b\| \geq 1$.

Proof Given a and b satisfying the hypothesis of the lemma, then

$$\begin{array}{rclcl} 0 & \neq & (a-b)^3 & = & (a-b)^5. \\ \text{Therefore } 0 & \neq & \|(a-b)^3\| & \leq & \|(a-b)^2\| \|(a-b)^3\|. \\ \text{Therefore} & & \|a-b\| & \geq & 1. \end{array}$$

□

5.3 Theorem (Isolation) The following statements are equivalent:

- (1) a is isolated in $L(A)$.
- (2) $a \in I(A) \cap Z(A)$ and any $b \in L(A)$ with $b^3 = a$ is such that $b = a$.

Proof Let $a \in L(A)$. Then the following conditions are mutually exclusive and exhaustive:

- (i) $a^2 \neq a$.
- (ii) $a^2 = a$ and $\exists b \in L(A)$ such that $a = b^3 = b^2 \neq b$.
- (iii) $a^2 = a$ and $\exists b \in L(A)$ such that $a = b^3 \neq b^2 = b$.
- (iv) $a^2 = a$ and every $b \in L(A)$ with $a = b^3$ is such that $b^3 = b^2 = b$.

For a to be isolated in $L(A)$; (i), (ii) and (iii) are not possible as then a will be arc connected in (i) to a^2 by the line segment $f_1|_{[0,1]}$ of the ray

$$f_1 := t \mapsto ta + (1-t)a^2 : [0, \infty) \rightarrow L(A),$$

in (ii) and in (iii) to b by the line segment $f_2|_{[0,1]}$ of the ray

$$f_2 := t \mapsto ta + (1-t)b : [0, \infty) \rightarrow L(A).$$

We are left with alternative (iv). Thus the conclusion here is that if a is isolated in $L(A)$ then

(iv)' $a \in I(A)$ and any $b \in L(A)$ with $b^3 = a$ is such that $b = a$.

Again let $a \in L(A)$. Then

$$\begin{aligned} F_a(\cdot, \cdot) &= (x, y) \mapsto a - xay - axay : A \times A \rightarrow A \\ F'_a(\cdot, \cdot) &:= (x, y) \mapsto a - xay - xaya : A \times A \rightarrow A \end{aligned}$$

are both continuous maps with $F_a(A \times A) \subset L(A)$, $F'_a(A \times A) \subset L(A)$ and

$$F_a(0, 0) = a = F'_a(0, 0).$$

If, in addition, a is isolated in $L(A)$, it follows that

$$(v) \quad axay = xay \quad \forall x, y \in A;$$

$$(vi) \quad xaya = xay \quad \forall x, y \in A.$$

Now manipulation of (iv)', (v) and (vi) shows that:

$$(a \text{ is isolated in } L(A)) \Rightarrow (a \in Z(A)).$$

Altogether therefore (1) \Rightarrow (2).

Conversely assume (2). Note that in (2)

$$[(b^3 = a) \Rightarrow (b = a)] \Leftrightarrow [(b \neq a) \Rightarrow (b^3 \neq a)].$$

Therefore along with Lemma 5.2,

$$(2) \Rightarrow (b \in L(A) \text{ with } b \neq a \text{ is such that } \|b - a\| \geq 1).$$

Hence a satisfying (2) is isolated in $L(A)$. \square

5.4 Theorem A component of $L(A)$ is either unbounded or a singleton.

Proof Theorem 5.3 characterizes the singleton components of $L(A)$. For $a \in L(A)$, let C_a be the component containing a . The conditions (i) – (iii) in the proof of Theorem 5.3 are mutually exclusive and exhaust the possibilities for C_a to be a non-singleton component.

Condition(i): With f_1 as defined in the proof of the the theorem, the entire ray $f_1([0, \infty))$ lie in C_a . Hence C_a is unbounded.

Conditions (ii) and (iii): Similarly the entire ray $f_2([0, \infty))$ lie in C_a under each of the conditions (ii) and (iii), where f_2 is also as defined in the proof of Theorem 5.3. Therefore C_a is also unbounded under each of the conditions (ii) and (iii). \square

5.5 Theorem A non-central or non-idempotent member of $L(A)$ is in some unbounded component of $L(A)$.

Proof The proof follows from the proof of Theorem 5.4. \square

5.6 Theorem The component of $L(A)$ containing the origin is a singleton iff A has no order. Otherwise it is unbounded.

Proof The theorem follows from Theorem 4.3 and Theorem 5.1(2). \square

5.7 Theorem (Existence of Isolated Points) $L(A)$ has an isolated point iff A has no order.

Proof Consider the *only if* part. Suppose $a \in L(A)$ is isolated. Then $a \in I(A) \cap L(A) \cap Z(A)$ and every $b \in L(A)$ with $b^2 = a^2$ is such that $b = a$. Now A is with order. Therefore $\exists c \in A$ such that $(cx = xc = 0 \quad \forall x \in A) \wedge (c \neq 0)$. Therefore $c \in L(A)$. Moreover $a + c \in L(A)$ since

$$\begin{aligned} (a + c)x(a + c)y &= (axa + axc + cxa + cxc)y \\ &= (ax + 0 + 0 + 0)y \\ &= axy + cxy \\ &= (a + c)xy \quad \forall x, y \in A. \end{aligned}$$

Now $(a + c)^2 = a^2$ but $a + c \neq a$ since $c \neq 0$. This is a contradiction. Therefore A is without order. As such $L(A)$ has an isolated point only if A has no order.

The *if* part follows from Theorem 4.3. \square

5.8 Corollary If A is a very nice Banach algebra, then $a \in L(A)$ is isolated iff a is central.

6 Connectedness and Endomorphic Left Elements; A , a Very Nice Banach Algebra

6.1 Theorem Let A be a very nice Banach algebra. Then the component of $L(A)$ containing an element $a \in L(A)$ coincide with the component of $I(A)$ containing a .

Proof Let $a \in L(A)$. Let K_a be the component of a in $I(A)$ and C_a the component of a in $L(A)$. Since $a \in L(A)$ and $K_a = \{\omega a \omega^{-1} : \omega \in G(\tilde{A})\}$, it follows from Theorem 3.1(4) that $K_a \subset L(A)$. Therefore $K_a \subset C_a$. Since $L(A) \subset I(A)$ by definition, we conclude that $K_a = C_a$. \square

Let A be a very nice Banach algebra. Having identified the components of $L(A)$ as those of $I(A)$, the following conclusions then follow from Zemanek's paper, [7]:

- (1) $(e, f \in L(A)) \wedge (r(e - f) < 1) \Rightarrow (e \text{ and } f \text{ are arc connected})$; where $r(x)$ is the spectral radius of x in A .
- (2) $L(A)$ is locally arc connected.
- (3) The singleton components of $L(A)$ are contained in the centre of A while the unbounded components are disjoint from this centre.
- (4) For two distinct components K_1, K_2 of $L(A)$; $d(K_1, K_2) \geq \rho(K_1, K_2) \geq 1$; where ρ is the spectral distance.
- (5) The distance of an unbounded component of $L(A)$ from the centre of A is at least $\frac{1}{2}$.

7 Example

$A = M_2(\triangle)$, the algebra of lower triangular 2×2 matrices.

$$L(A) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 1 & 0 \\ \alpha & 0 \end{pmatrix} : \alpha \in K \right\}.$$

$$R(A) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 0 & 0 \\ \alpha & 1 \end{pmatrix} : \alpha \in K \right\}.$$

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